Dieudonné module theory, part II: the classical theory

Alan Koch

Agnes Scott College

May 24, 2016

Outline

Pick a prime $p > 2$.

Recall: it is unreasonable to think that every finite abelian Hopf algebra H of p-power rank over an $\mathbb{F}_p$-algebra R can be classified with Dieudonné modules.

Last time, we focused on Hopf algebras with a nice coalgebra structure.

This time, we will make assumptions on both R and H.

The assumptions on R are very restrictive, and the ones on H are more palatable.

The situation du jour (et demain)

Throughout this talk, let $R = k$ be a perfect field of characterisic p.

Any finite k-Hopf algebra H can be written as

$$H \cong H_{r,r} \otimes H_{r,\ell} \otimes H_{\ell,r} \otimes H_{\ell,\ell}$$

where

- $H_{r,r}$ and $H^*_{r,r}$ are both reduced k-algebras.
- $H_{r,\ell}$ is a reduced and $H^*_{r,\ell}$ is a local k-algebra.
- $H_{\ell,r}$ is a local and $H^*_{r,\ell}$ is a reduced k-algebra.
- $H_{\ell,\ell}$ and $H^*_{\ell,\ell}$ are both local k-algebras.

If $p \nmid \dim_k H$ then $H = H_{r,r}$.

$$H \cong H_{r,r} \otimes H_{r,\ell} \otimes H_{\ell,r} \otimes H_{\ell,\ell}$$

Reduced k-Hopf algebras are “classified": they correspond to finite groups upon which $\mathrm{Gal}(\overline{k}/k)$ acts continuously.

Then $H_{r,r}$ and $H_{r,\ell}$ are classified, and by duality so is $H_{\ell,r}$.

Thus, the most mysterious Hopf algebras are the ones of the form $H_{\ell,\ell}$.

We will call these “local-local” and assume that all Hopf algebras in this talk are finite, abelian, and local-local over the perfect field k.

If H is local-local, then a result of Waterhouse is that H is a “truncated polynomial algebra”, i.e.,

$$H \cong k[t_1, \ldots, t_r]/(t_1^{p^{n_1}}, \ldots, t_r^{p^{n_r}}),$$

so the algebra structure isn’t terrible.

Suggested readings

- M. Demazure and P. Gabriel, Groupes Algébriques - Tome 1.
  Good news: partially translated into English as Introduction to Algebraic Geometry and Algebraic Groups.
  Bad news: not the part you need (chapter 5).
- A. Grothendieck, Groupes de Barsotti-Tate et Cristaux de Dieudonné.
  Explicit connection between Dieudonné modules and Hopf algebras.
- T. Oda, The first de Rham cohomology group and Dieudonné modules. The section on Dieudonné modules gave me my first explicit examples.
- R. Pink, Finite group schemes,
  https://people.math.ethz.ch/ pink/ftp/FGS/CompleteNotes.pdf.
  Good intro to Dieudonné modules with proofs (group scheme point of view).

The gist

What is a *k*-Hopf algebra?

It is a k-algebra and a k-coalgebra with compatible structures.

A Dieudonné module is a single module (over a ring TBD) which encodes both structures.

In Monday's talk, the coalgebra structure was assumed, and the algebra structure was encoded by the action of F on $D_*(H)$.

Outline

Witt polynomials

For each $n \in \mathbb{Z}^{\geq 0}$, define $\Phi_n \in \mathbb{Z}[Z_0, \dots, Z_n]$ by

$$\Phi_n(Z_0, \dots, Z_n) = Z_0^{p^n} + pZ_1^{p^{n-1}} + \cdots + p^n Z_n.$$

Note: Φ_n depends on the choice of prime p, which we assume to be "our" p.

Define polynomials $S_n, P_n \in \mathbb{Z}[X_0, \dots, X_n; Y_0, \dots, Y_n]$ implicitly by

$$\Phi_n(S_0, \dots, S_n) = \Phi_n(X_0, \dots, X_n) + \Phi_n(Y_0, \dots, Y_n)$$
$$\Phi_n(P_0, \dots, P_n) = \Phi_n(X_0, \dots, X_n)\Phi_n(Y_0, \dots, Y_n)$$

Fact. Yes, the coefficients are all integers.

$$\Phi_n(Z_0,\dots,Z_n) = Z_0^{p^n} + pZ_1^{p^{n-1}} + \cdots + p^n Z_n$$
$$\Phi_n(S_0,\dots,S_n) = \Phi_n(X_0,\dots,X_n) + \Phi_n(Y_0,\dots,Y_n)$$
$$\Phi_n(P_0,\dots,P_n) = \Phi_n(X_0,\dots,X_n)\Phi_n(Y_0,\dots,Y_n)$$

Example (low hanging fruit)

$$\Phi_0(Z_0) = Z_0$$
$$S_0(X_0;Y_0) = X_0 + Y_0$$
$$P_0(X_0,Y_0) = X_0 Y_0$$

Exercise 1. Prove:

$$S_1((X_0,X_1);(Y_0,Y_1)) = X_1 + Y_1 - \frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i} X_0^i Y_0^{p-i}$$
$$P_1((X_0,X_1);(Y_0,Y_1)) = X_0^p Y_1 + X_1^p Y_0 + pX_1 Y_1.$$

Let $W(\mathbb{Z}) = (w_0, w_1, w_2, \dots),\ w_i \in \mathbb{Z}$ for all i. Define addition and multiplication on $W(\mathbb{Z})$ by

$$\begin{aligned} (w_0, w_1, \dots) +_W (x_0, x_1, \dots) &= (S_0(w_0; x_0), S_1((w_0, w_1); (x_0, x_1)), \dots) \\ (w_0, w_1, \dots) \cdot_W (x_0, x_1, \dots) &= (P_0(w_0; x_0), P_1((w_0, x_1); (w_0, x_1)), \dots) \end{aligned}$$

These operations make $W(\mathbb{Z})$ into a commutative ring.

Actually, $W(-)$ is a $\mathbb{Z}$-*ring scheme*, meaning that for all $\mathbb{Z}$-algebras A, $W(A)$ is a ring.

In particular, set $W = W(k)$ and call this the ring of Witt vectors with coefficients in k (or "Witt vectors" for short).

Properties of Witt vectors. Show each of the following:

Exercise 2. W is a ring of characteristic zero.

Exercise 3. W is an integral domain.

Exercise 4. $(1, 0, 0, \ldots)$ is the multiplicative identity of W.

Exercise 5. $W(\mathbb{F}_p) \cong \mathbb{Z}_p$.

Exercise 6. $W(\mathbb{F}_{p^n})$ is the unramified extension of $\mathbb{Z}_p$ of degree n.

Exercise 7. The element $(0, 1, 0, 0 \ldots) \in W$ acts as mult. by p.

Exercise 8. $p^n W$ is an ideal of W.

Exercise 9. Let $W_n = (w_0, w_1, \ldots, w_{n-1})$. Then W_n is a ring with operations induced from W.

Exercise 10. $W/p^n W \cong W_n$. In particular, $W_0 \cong k$.

Of particular importance will be two operators F, V on W given by

$$F(w_0, w_1, w_2, \dots) = (w_0^p, w_1^p, w_2^p, \dots)$$
$$V(w_0, w_1, w_2, \dots) = (0, w_0, w_1, \dots)$$

F is called the *Frobenius* and V is called the *Verschiebung*.
Show each of the following:

Exercise 11. $FV = VF = p$ (multiplication by p).

Exercise 12. F, V both act freely on W, only F acts transitively.

Exercise 13. If $w \in p^n W$ then $Fw, Vw \in p^n W$. Thus, F and V make sense on W_n as well.

Exercise 14. W_n is annihilated by V^{n+1}.

Exercise 15. Any $w = (w_0, w_1, w_2, \dots) \in W$ decomposes as

$$(w_0, w_1, w_2, \dots) = \sum_{i=0}^{\infty} p^i (w_i^{p^{-i}}, 0, 0, \dots).$$

Outline

The Dieudonné ring

Now let w^σ be the Frobenius on $w \in W$ (invertible since k is perfect).

Let $E = W[F, V]$ be the ring of polynomials with

$$FV = VF = p, \; Fw = w^\sigma F, \; wV = Vw^\sigma; w \in W$$

We call E the *Dieudonné ring*.

Note that E is commutative if and only if $k = \mathbb{F}_p$.

One more construction

Let us view W_n as a group scheme (as opposed to the ring $W_n(k)$).

Let W_n^m be the m^{th} Frobenius kernel of the group scheme W_n, i.e. for any k-algebra A,

$$W_n^m(A) = \{(a_0, a_1, \ldots, a_{n-1}) : a_i \in A,\ a_i^{p^m} = 0 \text{ for all } 0 \le i \le n-1\}.$$

Some properties of W_n^m:

- Clearly, $F : W_n \to W_n$ restricts to $F : W_n^m \to W_n^m$.
- Also, $V : W_n \to W_n$ restricts to $V : W_n^m \to W_n^m$.
- For each k-algebra A, $W(k)$ acts on $W(A)$ through the algebra structure map $k \hookrightarrow A$.
- Every local-local Hopf algebra represents a subgroup scheme of some $(W_n^m)^r$.

There are maps $\iota : W_n^m \hookrightarrow W_n^{m+1}$ (inclusion) and $\nu : W_n^m \hookrightarrow W_{n+1}^m$ (induced by V) such that

$$\begin{array}{ccc} W_n^m & \xrightarrow{\iota} & W_n^{m+1} \\ \downarrow \nu & & \nu \downarrow \\ W_{n+1}^m & \xrightarrow{\iota} & W_{n+1}^{m+1} \end{array}$$

Then $\{W_n^m\}$ is a direct system (where the partial ordering $(n, m) \leq (n', m')$ is given by the conditions $n \leq n', \; m \leq m'$).

Let

$$\widehat{W} = \varinjlim_{m,n} W_n^m.$$

(Not standard notation.)

Let H be a local-local Hopf algebra. Define

$$D_*(H) = \text{Hom}_{k\text{-}gr}\left(\widehat{W}, \text{Spec}(H)\right).$$

The actions of F and V on $\widehat{W}$ induces actions on $D_*(H)$, as does the action of $W(k)$.

Thus, $D_*(H)$ is an E-module.

Furthermore, $D_*(-)$ is a covariant functor, despite the fact this is referred to as "contravariant Dieudonné module theory".

$$D_*(H) = \mathrm{Hom}_{k\text{-}gr}\left(\widehat{W}, \mathrm{Spec}(H)\right).$$

Properties of D_*:

- $D_*(H)$ is killed by some power of F and V.
- $D_*(H)$ is finite length as a W-module.
- $\mathrm{length}_W D_*(H) = \log_p \dim_k H$.
- $D_*(H_1 \otimes H_2) \cong D_*(H_1) \times D_*(H_2)$.
  Note $\mathrm{Spec}(H_1 \otimes H_2) \cong \mathrm{Spec}(H_1) \times \mathrm{Spec}(H_2)$.

Theorem (Main result in Dieudonné module theory)

D_* induces a categorical equivalence

$$\left\{ \begin{array}{c} p\text{- power rank} \\ \text{local-local} \\ k\text{-Hopf algebras} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} E\text{-modules} \\ \text{of finite length over } W(k) \\ \text{killed by a power of } F \text{ and } V \end{array} \right\}.$$

We call E-modules satisfying the above *Dieudonné modules*.

Legal disclaimer

$$\left\{ \begin{array}{c} E\text{-modules} \\ \text{of finite length over } W(k) \\ \text{killed by a power of } F \text{ and } V \end{array} \right\} = \{\text{"Dieudonné modules"}\}$$

This is not everyone's definition.

Dieudonné modules can also be used to describe

- p-power rank Hopf algebras (local, separable).
- p-divisible groups
- graded Hopf algebras
- primitively generated Hopf algebras
- a slew of other things

Another legal disclaimer

$$\left\{ \begin{array}{c} E\text{-modules} \\ \text{of finite length over } W(k) \\ \text{killed by a power of } F \text{ and } V \end{array} \right\} = \{\text{“Dieudonné modules”}\}$$

Some literature will describe a Dieudonné module as a triple (M, F, V) where

- M is a finite length $W(k)$-module
- $F : M \to M$ is a nilpotent σ-semilinear map
- $V : M \to M$ is a nilpotent σ^{-1}-semilinear map.

This is just a different way to describe the same thing.

A Hopf algebra interpretation?

$$D_*(H) = \mathrm{Hom}_{k\text{-}gr}\left(\widehat{W}, \mathrm{Spec}(H)\right).$$

Each W_n^m is an affine group scheme, represented by

$$H_{m,n} = k[t_0, \ldots, t_{n-1}]/(t_1^{p^m}, t_2^{p^m}, \ldots, t_{n-1}^{p^m})$$

with comultiplications induced from Witt vector addition:

$$\Delta(t_i) = S_i((t_0 \otimes 1, \ldots, t_i \otimes 1); (1 \otimes t_0, \ldots, 1 \otimes t_i)).$$

In theory we could possibly write something like

$$D_*(H) = \mathrm{Hom}_{k\text{-Hopf}}(H, \varprojlim_{m,n} H_{m,n}).$$

However, I have never found this to be helpful.

No wait, come back

Suppose we are given a Dieudonné module. What is the corresponding Hopf algebra?

Suppose $V^{N+1}M = 0$. Let $H = k[\{T_m : m \in M\}]$ subject to the relations

$$\begin{aligned} T_{Fm} &= (T_m)^p \\ T_{m_1+m_2} &= S_N((T_{V^N m_1}, \dots, T_{Vm_1}, T_{m_1}); (T_{V^N m_2}, \dots, T_{Vm_2}, T_{m_2}) \\ T_{(w_0,w_1,\dots)m} &= P_N((w_0^{p^{-N}}, \dots, w_N^{p^{-N}}); (T_{V^N m}, \dots, T_{Vm}, T_m), \\ m, m_1, m_2 &\in M, \ (w_0, w_1, \dots) \in W. \end{aligned}$$

Define $\Delta : H \to H \otimes H$ by

$$\Delta(T_m) = S_N((T_{V^N m} \otimes 1, T_{V^{N-1} m} \otimes 1, \dots, T_m \otimes 1); (1 \otimes T_{V^N m}, \dots, 1 \otimes T_m)).$$

Then H is a local-local k-Hopf algebra.

The gist, part 2

$$T_{Fm} = (T_m)^p$$
$$T_{m_1+m_2} = S_N((T_{V^N m_1}, \ldots, T_{Vm_1}, T_{m_1}); (T_{V^N m_2}, \ldots, T_{Vm_2}, T_{m_2})$$
$$T_{wm} = P_N((w_0^{p^{-N}}, \ldots, w_N^{p^{-N}}); (T_{V^N m}, \ldots, T_{Vm}, T_m)$$
$$\Delta(T_m) = S_N((T_{V^N m} \otimes 1, \ldots, T_m \otimes 1); (1 \otimes T_{V^N m}, \ldots, 1 \otimes T_m)).$$

The action of F describes the (interesting) algebra structure.

The action of V describes the coalgebra structure.

Exercise 16. Show that the N above need not be minimal.

Exercise 17. Let M be a Dieudonné module. Show that $T_0 = 0$.

Outline

$$\begin{aligned} T_{Fm} &= (T_m)^p \\ T_{m_1+m_2} &= S_N((T_{V^N m_1}, \ldots, T_{Vm_1}, T_{m_1}); (T_{V^N m_2}, \ldots, T_{Vm_2}, T_{m_2}) \\ T_{wm} &= P_N((w_0^{p^{-N}}, \ldots, w_N^{p^{-N}}); (T_{V^N m}, \ldots, T_{Vm}, T_m) \\ \Delta(T_m) &= S_N((T_{V^N m} \otimes 1, \ldots, T_m \otimes 1); (1 \otimes T_{V^N m}, \ldots, 1 \otimes T_m)). \end{aligned}$$

Example

The simplest possible (nontrivial) E-module is a k-vector space M of dimension 1 with F and V acting trivially.
In other words, $M = E/E(F, V)$. In particular, $N = 1$. Let M have k-basis $\{x\}$, and let H be the Hopf algebra with $D_*(H) = M$.

Exercise 18. Prove H is generated as a k-algebra by t, where $t = T_x$.

Since

$$t^p = \left(T_x\right)^p = T_{Fx} = T_0 = 0,$$

we have $H = k[t]/(t^p)$. Also, by the formulas above, t is primitive.

Primitively generated flashback

Suppose H is primitively generated (and local-local), and let $M = D_*(H)$.

Since

$$\Delta(T_m) = S_N((T_{V^N m} \otimes 1, \ldots, T_m \otimes 1); (1 \otimes T_{V^N m}, \ldots, 1 \otimes T_m))$$

it follows that we may take $N = 0$.

Thus $VM = 0$, hence M can be viewed as a module over $k[F]$ with

$$T_{Fm} = (T_m)^p$$
$$T_{m_1+m_2} = T_{m_1} + T_{m_2}$$
$$T_{wm} = w_0 T_m.$$

In this case, it's the same module as yesterday.

Example

Let $M = E/(F^m, V^n) = D_*(H)$.

Exercise 19. What is pM?

Exercise 20. Exhibit a k-basis for M.

Exercise 21. Prove that

$$H = k[t_1, \ldots, t_n]/(t_1^{p^m}, \ldots, t_n^{p^m})$$

Exercise 22. Write out the comultiplication (in terms of Witt vector addition).

Exercise 23. Show that $\mathrm{Spec}(H) = W_n^m$.

Let $H = k[t]/(t^{p^5})$ with

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t^{p^3 i} \otimes t_1^{p^3(p-i)}.$$

What is $D_*(H)$?
Let $M = D_*(H)$, and suppose there is an $x \in M$ such that $t = T_x$. Then

$$0 = t^{p^5} = (T_x)^{p^5} = T_{F^5 x}$$

so $F^5 x = 0$, from which it follows that $F^5 M = 0$ but $F^4 M \neq 0$. Also,

$$\begin{aligned}\Delta(t) &= S_1\left(((T_x)^{p^3} \otimes 1, T_x \otimes 1); (1 \otimes (T_x)^{p^3}, 1 \otimes T_x)\right) \\ &= S_1\left(T_{F^3 x} \otimes 1, T_x \otimes 1); (1 \otimes T_{F^3 x}, 1 \otimes T_x\right)\end{aligned}$$

Then $N = 1$ (i.e., $V^2 M = 0$) and $Vm = F^3 m$. Thus,

$$M \cong E/E(F^5, F^3 - V, V^2) = E/E(F^5, F^3 - V).$$

This can be confirmed by working backwards.

Let $H = k[t_1, t_2]/(t_1^{p^2}, t_2^{p^2})$ with

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1$$

$$\Delta(t_1) = t_2 \otimes 1 + 1 \otimes t_2 + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t_1^{pi} \otimes t_1^{p(p-i)}.$$

What is $D_*(H)$?
Let $M = D_*(H)$, and suppose $x, y \in M$ satisfy $t_1 = T_x,\ t_2 = T_y$.
Clearly, $F^2x = F^2y = 0$ and

$$\begin{aligned}
\Delta(t_1) &= S_0(T_x \otimes 1; 1 \otimes T_x) \\
&= S_1((0 \otimes 1, T_x \otimes 1); (1 \otimes 0, 1 \otimes T_x)) \\
\Delta(t_2) &= S_1\big(((T_x)^p \otimes 1, T_y \otimes 1); (1 \otimes (T_x)^p, 1 \otimes T_y)\big) \\
&= S_1\big((T_{Fx} \otimes 1, T_y \otimes 1); (1 \otimes T_{Fx}, 1 \otimes T_y)\big).
\end{aligned}$$

Thus $Vx = 0$ and $Vy = Fx$. So,

$$M = Ex + Ey,\ F^2M = V^2M = 0, Vx = 0, Fx = Vy.$$

Exercises!

Write out the Hopf algebra for each of the following. Be as explicit as you can.

Exercise 24. $M = E/(F^2, F - V)$

Exercise 25. $M = E/E(F^2, V^2)$

Exercise 26. $M = E/E(F^2, p, V^2)$

Exercise 27.
$M = Ex + Ey,\; F^4x = 0, F^3y = 0, Vx = F^2y, V^2x = 0, Vy = 0.$

More exercises!

Find the Dieudonné module for each of the following.

Exercise 28. $H = k[t]/(t^{p^4}),\ t$ primitive.

Exercise 29. $H = k[t_1, t_2]/(t_1^{p^3}, t_2^{p^2}),\ t_1$ primitive and

$$\Delta(t_2) = t_2 \otimes 1 + 1 \otimes t_2 + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t_1^{pi} \otimes t_1^{p(p-i)}.$$

Exercise 30. $H = k[t_1, t_2]/(t_1^{p^3}, t_2^{p^2}),\ t_1$ primitive and

$$\Delta(t_2) = t_2 \otimes 1 + 1 \otimes t_2 + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t_1^{p^2 i} \otimes t_1^{p^2(p-i)}.$$

Outline

How does duality work?

Very nicely, with the proper definitions.

For an E-module M we define its dual M^* to be

$$M^* = \mathrm{Hom}_W(M, W[p^{-1}]/W)$$

This is an E-module, where F and V act on $\phi : M \to W[p^{-1}]/W$ as follows:

$$(F\phi)(m) = (\phi(Vm))^\sigma$$
$$(V\phi)(m) = (\phi(Fm))^{\sigma^{-1}}$$

Note. The $\sigma,\ \sigma^{-1}$ make $F\phi,\ V\phi$ into maps which are W-linear.

$M^* = \mathrm{Hom}_W(M, W[p^{-1}]/W)$

Properties

- If M is a Dieudonné module, so is M^*.
- $(E/E(F^n, V^m))^* = E/E(F^m, V^n)$.
  "Duality interchanges the roles of F and V."
- $D_*(H^*) = (D_*(H))^*$.

An example

Let $H = \mathbb{F}_p[t]/(t^{p^3})$ with

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t^{p^2 i} \otimes t^{p^2(p-i)}.$$

The choice of $k = \mathbb{F}_p$ is only for ease of notation.

Then $M = E/E(F^3, F^2 - V) = kx \oplus kFx \oplus kF^2 x$ as W-modules.

We wish to compute $\mathrm{Hom}_W(M, W[p^{-1}]/W) = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$.

$\mathrm{Hom}_{\mathbb{Z}_p}(E/E(F^3, F^2 - V), \mathbb{Q}_p/\mathbb{Z}_p)$

Note that any $\phi : kx \oplus kFx \oplus kF^2x \to \mathbb{Q}_p/\mathbb{Z}_p$ is determined by the images of $x,\ Fx$, and F^2x.

Note also that $V(Fx) = V(F^2x) = 0$, which will be useful later.

Since $px = FVx = F(F^2x) = F^3x = 0$,

$$M^* \cong \{(w, y, z) \in (\mathbb{Q}_p/\mathbb{Z}_p)^3 : pw, py, pz \in \mathbb{Z}_p\}$$

where $\phi \in M^*$ is given by

$$\begin{aligned} \phi(x) &= w \\ \phi(Fx) &= y \\ \phi(F^2x) &= z \end{aligned}$$

$\phi(x) = w,\ \phi(Fx) = y,\ \phi(F^2x) = z$

$$M^* \cong \{(w, y, z) \in (\mathbb{Q}_p/\mathbb{Z}_p)^3 : pw, py, pz \in \mathbb{Z}_p\}$$

Now F acts on ϕ by

$$(F\phi)(x) = (\phi(Vx))^\sigma = (\phi(F^2x))^\sigma = z^\sigma = z,\ (F\phi)(y) = (F\phi(z)) = 0$$

and V acts on ϕ by

$$\begin{aligned} (V\phi)(x) &= (\phi(Fx))^{\sigma^{-1}} = y^{\sigma^{-1}} = y \\ (V\phi)(Fx) &= (\phi(F^2x))^{\sigma^{-1}} = z^{\sigma^{-1}} = z \\ (V\phi(z)) &= 0. \end{aligned}$$

Thus, $F(w, y, z) = (z, 0, 0),\ V(w, y, z) = (y, z, 0)$.
Clearly, for all $u = (w, y, z)$ we have $V^3u = 0$ and $V^2u = Fu$.
It follows that $M^* = E/E(V^3, V^2 - F)$.

Exercise 31. Using M^*, give the algebra structure for H^*. (**Hint.** It requires two generators.)

Exercise 32. Using M^*, give the coalgebra structure for H^*.

Exercise 33. What changes if $k \neq \mathbb{F}_p$?

Outline

More is coming tomorrow.

Thank you.